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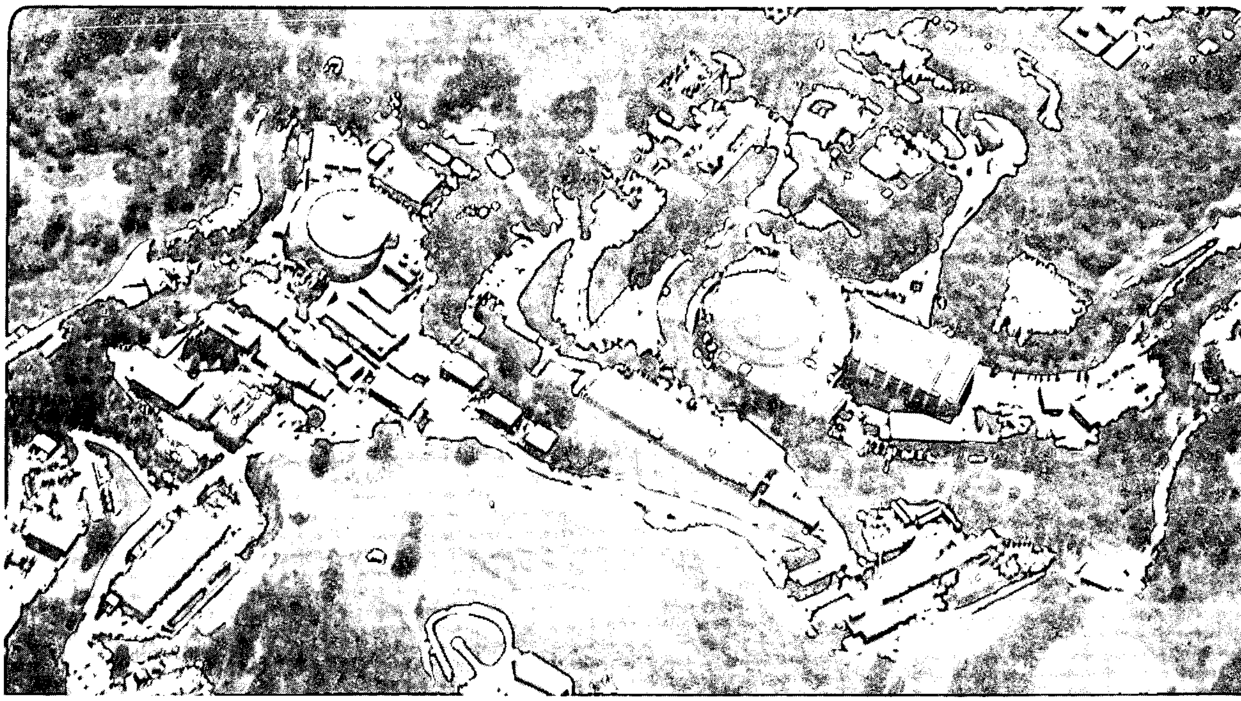

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UNIVERSAL DECAY OF VORTEX DENSITY IN TWO DIMENSIONS

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Universal decay of vortex density in two dimensions

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Recent numerical and experimental results for freely-decaying two-dimensional turbulence point to a universal decay in vortex density, $\rho \sim t^{-\xi}$ ($\xi \simeq 0.73$), from highly turbulent to laminar, vortex-free flows. This vortex decay has been successfully described in terms of a point-vortex dynamics, in close resemblance to the vortex dynamics for the two-dimensional Ginzburg-Landau equation. Here, we add numerical evidence and give theoretical arguments that the vortex density at late times follows the conjectured power law with $\xi = 3/4$.

1. Introduction

Vortices are an essential part of turbulence. An understanding of their dynamics and interactions is likewise crucial to the description of superconductors and superfluids. Recently, the role of vortices in the free decay of two-dimensional turbulence has come under close scrutiny [1-4]. The evolution of vorticity is given by the fluid-dynamical equation

$$\partial_t \omega + J(\psi, \omega) = \nu \nabla^2 \omega, \quad \omega \equiv \nabla^2 \psi, \qquad (1a, b)$$

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where ω is the scalar vorticity, ψ the stream function, ν the kinematic viscosity, and $J(\psi,\omega)=\partial_x\psi\partial_y\omega-\partial_x\omega\partial_y\psi$ the Jacobian. Through numerical integrations of this system, McWilliams [1] found that the vortex density ρ decays in time according to a power law,

$$\rho \sim t^{-\xi}, \tag{2}$$

with a non-integral exponent $\xi \simeq 0.71$. In later works [2,3] the value is variously reported as $\xi \simeq 0.72-0.75$. Furthermore, a recent turbulence experiment by Tabeling *et al.* [4] yielded $\xi = 0.7 \pm 0.1$, in support of universality.

Preliminary results have pointed to a universal ξ value for a whole hierarchy of freely-decaying turbulence models [2]. These models are based on a Hamiltonian point-vortex dynamics, which has been shown to be an excellent approximation [5]. In these models, the vortices are replaced by a particle placed at the vortex center. Each particle has a 'charge' equal to the total integrated vorticity $\oint \omega d\mathbf{x}$ of the spread vortex, and moves in a logarithmic potential defined by the harmonic stream function. Moreover, two particles with 'charges' of equal sign merge when they get closer than a given distance. Based on the above 'vortex census' [1], we add here numerical evidence and give general arguments to show that the vortex density $\rho(t)$ (at late times) decays according to a power law with a universal exponent $\xi = 3/4$.

2. Vortices in the Ginzburg-Landau equation

The point-vortex picture for two-dimensional turbulent flows has a strong resemblance to the Coulomb gas (where charges of opposite sign merge). We have, therefore, undertaken a large-scale numerical study of the vortex decay in the two-dimensional Ginzburg-Landau equation, which is the prototypical equation relevant to the description of vortices in Coulomb-gas systems like the XY model and thin superconducting films (6). The noiseless (zero-temperature), time-dependent Ginzburg-Landau equation takes the form

$$\dot{A}(\mathbf{x},t) = \mu A(\mathbf{x},t) - |A(\mathbf{x},t)|^2 A(\mathbf{x},t) + \nabla^2 A(\mathbf{x},t), \tag{3}$$

where $A(\mathbf{x}, t)$ is a complex order parameter at point $\mathbf{x}$ and time t. In the context of fluid dynamics, the Ginzburg-Landau equation is also refered to as the Stewartson-Stuart equation [7,8. For $\mu \leq 0$, $A(\mathbf{x}, t) = 0$; for $\mu > 0$, $A = \sqrt{\mu}$ is a homogeneous, stable solution. However, the decay towards the homogeneous state from an initial random (high-T) state with many vortices is far from trivial. As we shall see, our numerical results give further support for universality.

A point-vortex picture applies also to the Ginzburg-Landau system. In comparison with the point-vortex picture for turbulence, we write $A = \exp(\psi + i\phi)$, where ψ corresponds to the stream function and the phase ϕ is the dual potential. Moreover, the 'charges' have integral values, which quickly reduce to $+1$ (vortex) or -1 (antivortex). Thus, merging here means annihilation. By definition, the circulation of the gradient of the phase over a closed loop containing n_+ vortices and n_- antivortices is $\Delta\phi = \oint d\phi = 2\pi(n_+ - n_-)$. The phase $\phi(\mathbf{x}, t)$ at a vortex center is undefined, but $|A| = \exp(\psi) \to 0$, so A remains well-defined.

3. Simulations

The numerical simulation of the Ginzburg-Landau equation is sped up substantially by separating the local Stuart-Landau part (the first two terms) from the non-local diffusive term [9,10]. The iterative coupled-map system thus obtained is

$$A(\mathbf{x}, t + 2\tau) = F[e^{\tau\nabla^2} A(\mathbf{x}, t)], \tag{4}$$

where

$$F(z) \equiv z/\sqrt{\lambda + (1-\lambda)|z|^2/\mu}, \quad \lambda = e^{-2\mu\tau}, \tag{5a, b}$$

is the solution to the Stuart-Landau equation. Moreover, we use a square lattice, invoking the approximations

$$e^{\tau\nabla^2} \simeq (1 + \tau\nabla^2/m)^m \tag{6a}$$

and

$$\nabla^2 A(\mathbf{x}, t) = \frac{1}{4}\sum_{\mathbf{x}'}[A(\mathbf{x}', t) - A(\mathbf{x}, t)], \tag{6b}$$

where $m = 5$, and the sum is over the four nearest neighbors $\mathbf{x}'$. Larger values of m had no appreciable effect on the observed dynamics. The algorithm takes advantage of the parallel architecture of the Connection Machine 2.

Fig. 1 shows the evolution of the vortex density ρ for $\mu = 1$ and $\tau = 0.2$. The system is initialized in a random state on a lattice of size 512×512, with periodic boundary conditions. Initially, a state with many moving vortices and antivortices is formed (fig. 2a), but in time, the vortices and antivortices meet and annihilate; $n_+ = n_-$ for the system as a whole (conservation of vorticity). At early times, the rate of annihilation is proportional to the density squared, giving the value [11,12] $\xi = 1$, but at later times the vortices are more sparsely distributed (fig. 2b), making the annihilation rate non-trivial. Here we find numerically that the vortex decay follows a power law, with $\xi = 0.74 \pm 0.02$ (fig. 1). This result agrees with simulations for the scattering function by Mondello and Goldenfeld [13] (who however speculate that $\xi = 1$ at very late stages). There is thus strong evidence for a universal vortex-decay law. We now present a theory for this exponent.

4. Dynamical length scale

A length scale R, characteristic of time t, can be extracted by considering the average initial separation between two vortices that merge (annihilate) at time t. At late times (low density), the vortices interact via a logarithmic pair potential $\Psi(\mathbf{x}_i, \mathbf{x}_j) = -k \log(r)$, where $r = |\mathbf{x}_i - \mathbf{x}_j|$. The dissipative equation of motion, $\dot{r} = -\frac{\partial \Psi}{\partial r}$ (or in the hydrodynamical case, $\dot{r} = -\frac{\partial \Phi}{\partial r}$, where Φ is the phase field dual to Ψ), relates the initial separation R between two merging vortices to their merging (annihilation) time t,

$$R \sim \sqrt{t}. \tag{7}$$

Computations of the annihilation times for our Ginzburg-Landau system of many vortices on a 128×128 lattice are in agreement with this square-root law (fig. 3).

The theory of continuous ordering [14] provides another reason to expect a dynamical length scale $R(t)$ that scales like $\sqrt{t}$. The vortex dynamics can be viewed as the nonequilibrium phase ordering arising from an instantaneous quench from a high-temperature to a zero-temperature configuration. One could arbitrarily divide the phase $\phi(\mathbf{x}, t)$ of A into two states by assigning $+1$ to sites with $Im(A) > 0$, and -1 to sites with $Im(A) < 0$ (in the hydrodynamical case, ϕ is the potential dual to the stream function ψ). In this way, the complex order parameter is transformed to an up-down spin. In figs. 4a–f we show this two-state phase at six different times, each frame a factor of four later in time than the previous frame. State $+1$ appears as yellow, state -1 as blue. The vortices and antivortices lie on the $+1/-1$ interface. With no locally-conserved order parameter, the square-root law is predicted by the Lifshitz-Allen-Cahn theory [14,15].

5. Perimeter law

The decay can be understood in terms of the dynamical length $R(t)$, which we associate with an interaction volume [16,17]: After a time t, all vortex-antivortex pairs within an area $\mathcal{A} \sim R^2(t)$ have had a chance to annihilate. The number of remaining vortices $\rho(t)\mathcal{A}$ equals $\delta(\Delta\phi_{\mathcal{A}})/2\pi$, where $\delta(\Delta\phi_{\mathcal{A}})$ is the standard deviation of the circulation $\Delta\phi$ over a perimeter enclosing the area $\mathcal{A}$. One might reason [18] that because of the presence of free vortices in the initial high-temperature regime, the variance of $\Delta\phi_{\mathcal{A}}$ should scale like the area of $\mathcal{A}$. Though intuitively plausible, this 'area law' is incorrect. Based on a (high-temperature) uniform distribution of mutually-independent phase differences (mod 2π) between neighboring sites, Dhar [19] showed that, for large areas (late times), the variance of $\Delta\phi_{\mathcal{A}}$ at all temperatures is proportional to the perimeter of $\mathcal{A}$, i.e. $\delta^2(\Delta\phi_{\mathcal{A}}) \sim R$. (In fact, this result is a consequence of Stokes' law [13].) Thus,

$$\rho R^2 \sim \delta(\Delta\phi_{\mathcal{A}}) \sim \sqrt{R}. \tag{8}$$

In other words, $\rho \sim R^{-3/2}$, implying the exponent $\xi = 3/4$.

We point out that the initial separation $R \sim \rho^{-2/3}$ between vortices that later merge (annihilate) differs from the average distance $l \sim \rho^{-1/2}$ between vortices (of both signs). It is a consequence of this, and not of anomalous diffusion, that close encounters between vortex pairs occur less frequently than expected from the random motion of (sign-less) vortices [1].

In conclusion, we have presented numerical evidence and general arguments to show that the freely-decaying vortex density in two-dimensional Coulomb-gas systems, including freely-decaying two-dimensional turbulence, follows a power law at late times with a universal exponent $\xi = 3/4$.

Acknowledgements

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FIGURE CAPTIONS

Figure 1: Log-log plot of vortex density $\rho(t)$ obtained from the Ginzburg-Landau equation (3) with $\mu = 0.2$ and $\tau = 0.2$, starting from a random initial configuration. Initial slope is compared with a slope of -1. The asymptotic slope is found to be $\xi = 0.74 \pm 0.02$.

Figure 2: Phase $\phi(\mathbf{x}, t)$ of the order parameter A for the Ginzburg-Landau equation with $\mu = 0.2$ and $\tau = 0.2$. (a) $t = 64$. (b) $t = 4096$. The color change from blue to red corresponds to a change in phase from $-\pi$ to π (lattice size 512×512).

Figure 3: Log-log plot of vortex-antivortex separation R (at time $t = 30$) versus annihilation time t. The solid line has slope $1/2$.

Figure 4: Phase coarsening. The phase $\phi(\mathbf{x}, t)$ (as in fig. 2) is divided into two states by assigning $+1$ (yellow) to sites with $Im(A) > 0$, and -1 (blue) to sites with $Im(A) < 0$. The vortices and antivortices must lie on the $+1/-1$ interface. (a) $t = 4$. (b) $t = 16$. (c) $t = 64$. (d) $t = 256$. (e) $t = 1024$. (f) $t = 4096$.

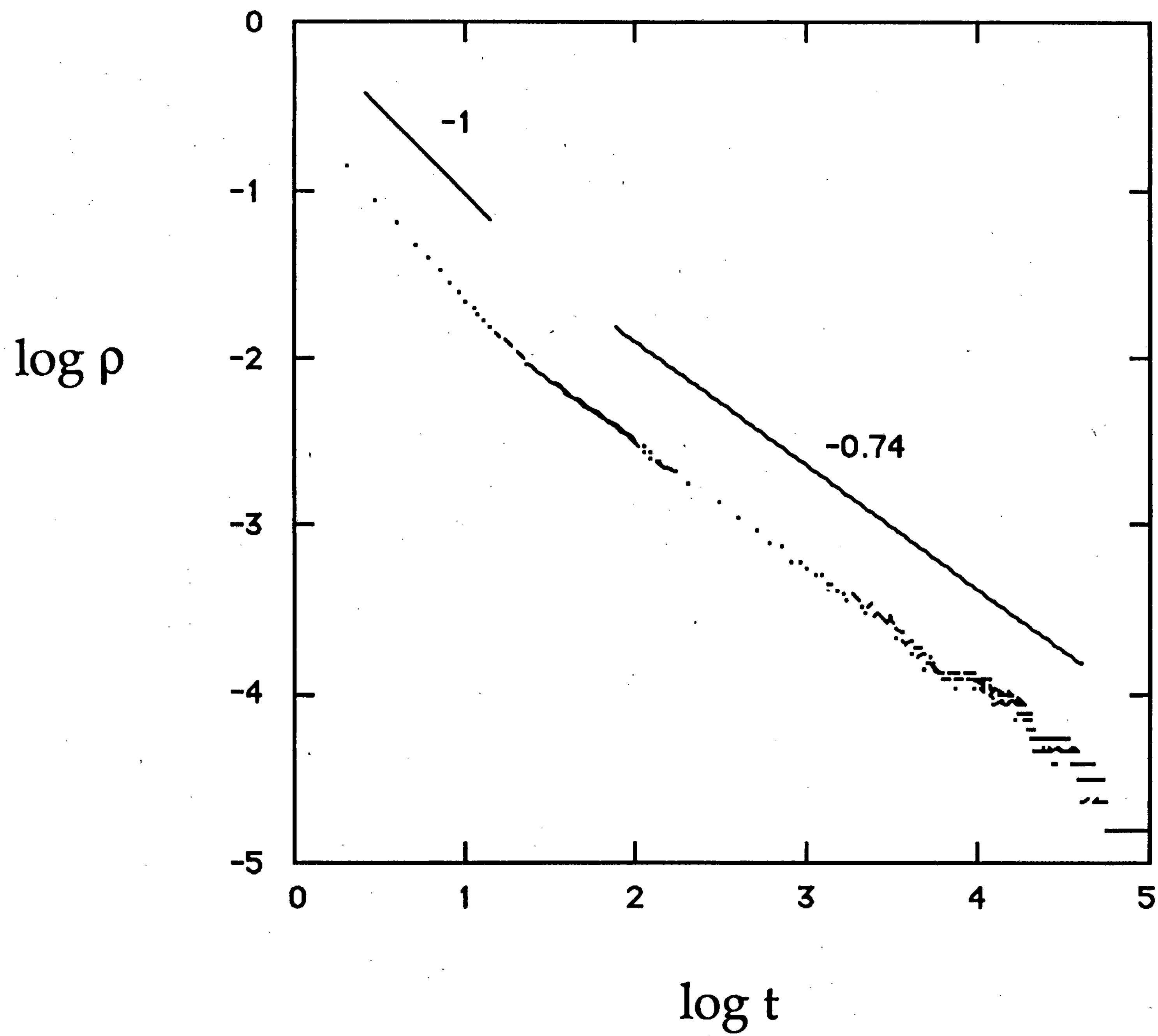

Figure 1

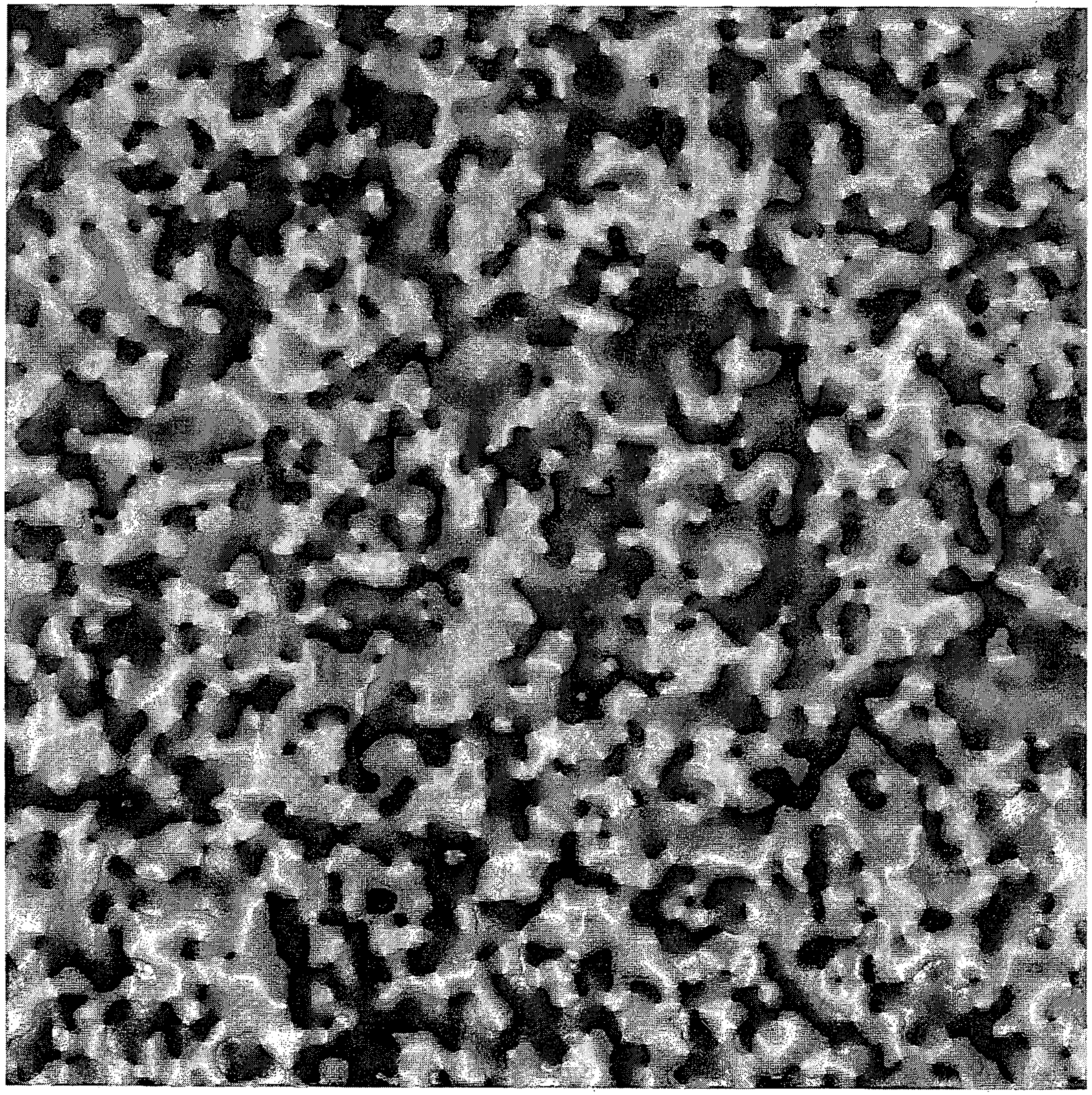

Figure 2 (a)

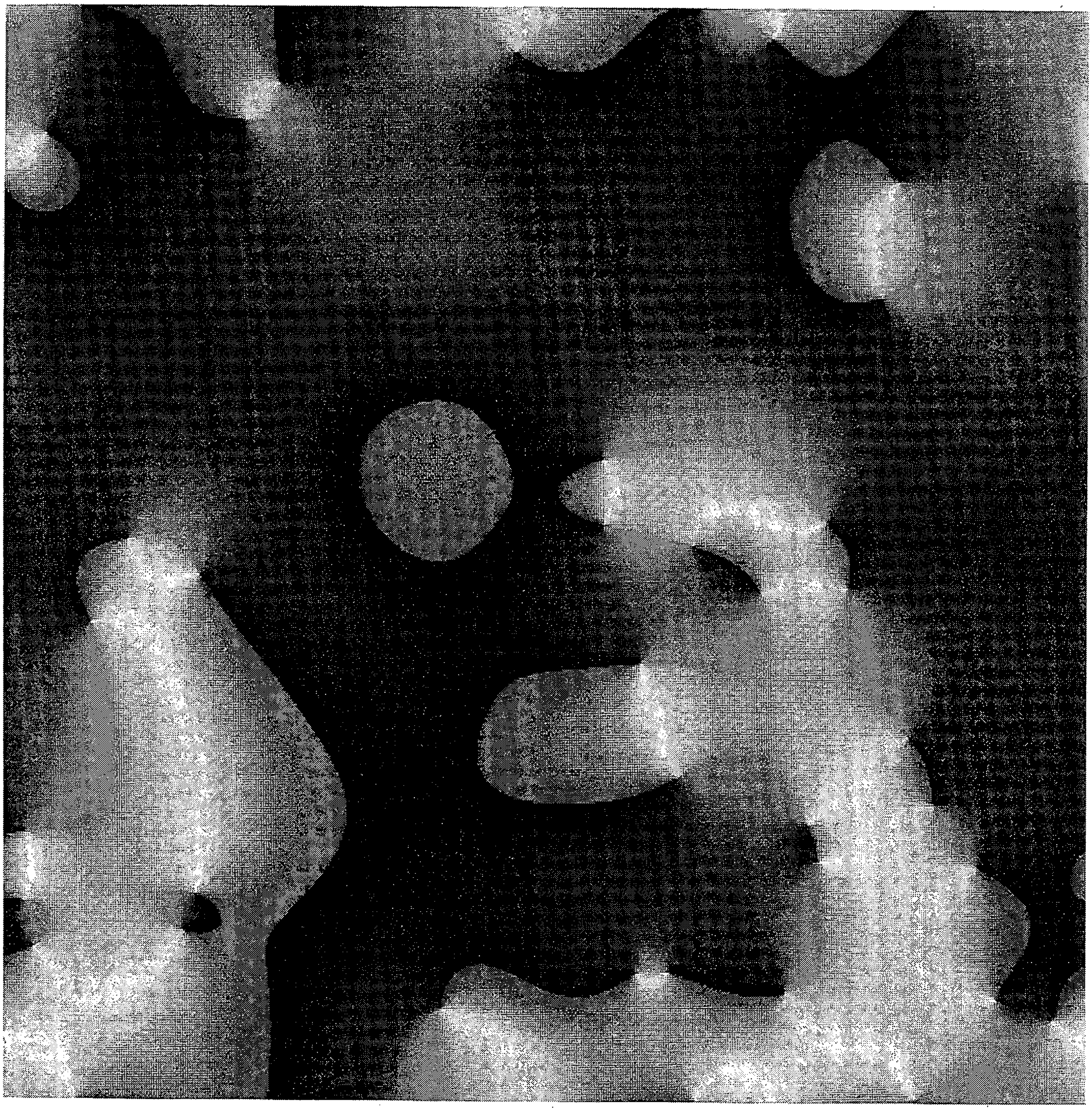

Figure 2 (b)

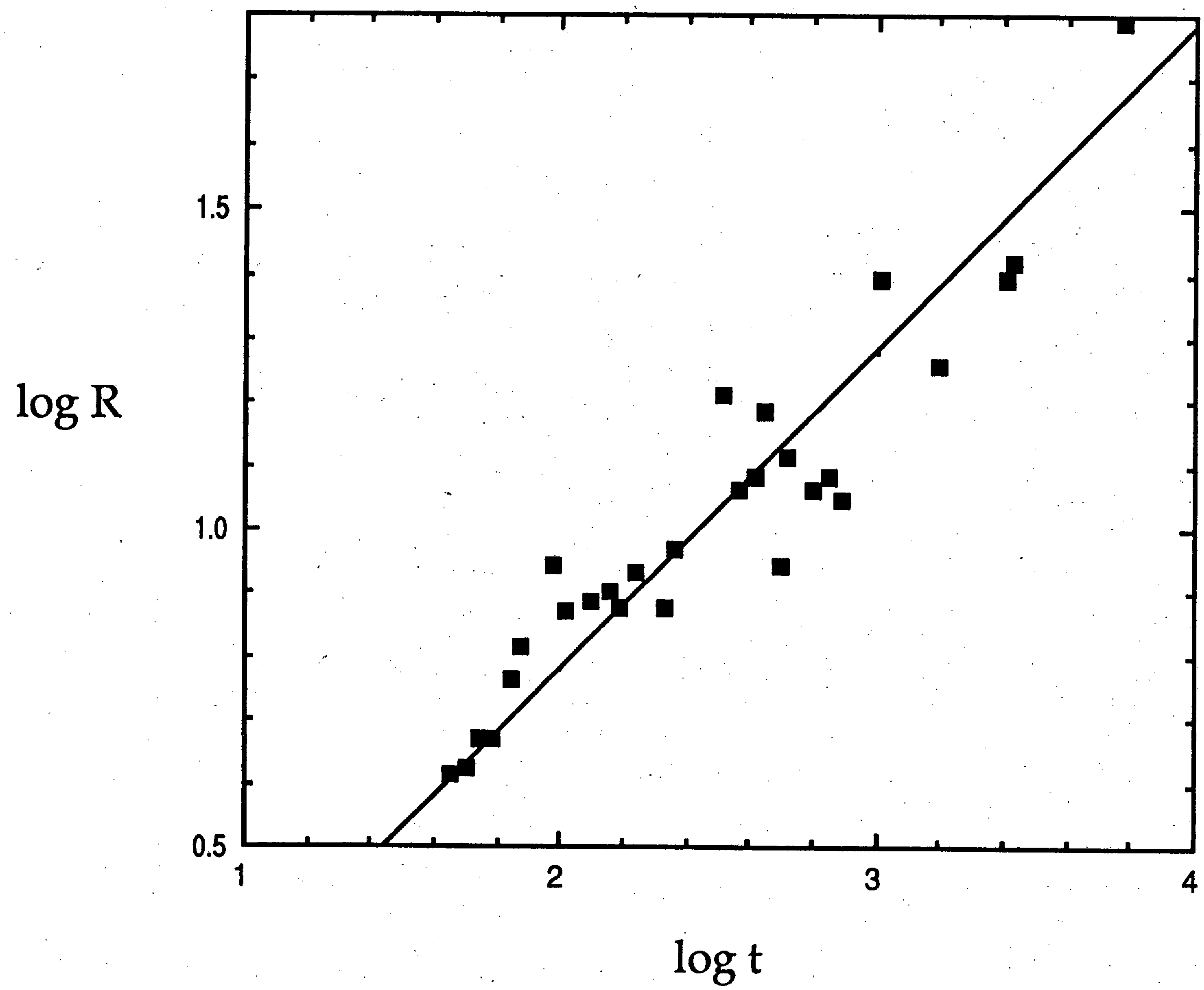

Figure 3

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